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STRUCTURAL INSTABILITY OF EXPONENTIAL FUNCTIONS

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ABSTRACT. We first prove some equivalent statements on J-stability of families of critically finite entire functions. Then, with these in hand, a conjecture concerning stability of the family of exponential functions is affirmatively answered in some cases.

0. INTRODUCTION

In recent years, there have been many papers on the dynamics of some important classes of transcendental entire functions. The Julia set of a transcendental entire function is necessarily quite different from that of a polynomial or a rational map since the point at infinity is an essential singularity of transcendental entire function, although many results are true for both rational functions and transcendental entire functions.

One of the basic problems in iteration theory is to study structural stability of holomorphic families (for definitions, see §1). The simplest and the most interesting family is the exponential function family. A well-known conjecture (e.g. see [6]) in this field is that λe^z $(\lambda \neq 0)$ is not structurally stable. In 1984, R. Devaney [4] proved that e^z is not structurally stable. In 1989, J. Zhou and Z. Li [11] proved that λe^z is unstable for any $\lambda > 1/e$. Here we show that λe^z is not structurally stable if λ is on Cantor sets of curves tending to ∞ in the λ-plane, which is called the "Hairs" [3], (where the half line $\lambda > 1/e$ in the λ-plane is contained in the "hairs"), and other choices of λ. More interesting, the bifurcation diagram of λe^z in the λ-plane is known as Figure 1 (e.g. [1, 3, and 5]). Roughly speaking, for any λ in the black area of Figure 1, we have $J(E_\lambda) = \mathbb{C}$ and for any λ in the white area in Figure 1, the $J(E_\lambda) \neq \mathbb{C}$. The black regions correspond to curves where the Julia set is the whole plane. The computer algorithm used to generate this picture simply colored a point if the corresponding exponential map satisfied $\operatorname{Re} E_\lambda^n(0) > 100$ for some $n \leq 100$. However the main result in this paper shows that for any black λ, which lies on "hairs", there is a white λ which can be arbitrarily close to the black λ. Of course, we cannot see this from the computer picture.

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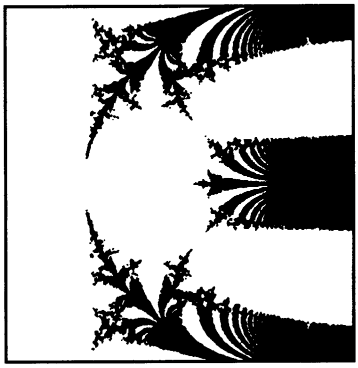

FIGURE 1 [3][1]. The topological structure of the bifurcation diagram of λe^z.

1. NOTATIONS AND PRELIMINARIES

In the sequel we always regard f as a transcendental entire function and denote the n-fold iterate of f by f^n. Set

$$F(f) = \{z: \{f^n\} \text{ is normal in a neighborhood of } z\}.$$

Then the Julia set of f is $J(f) = \mathbb{C}\backslash F(f)$. A point $a \in \mathbb{C}$ is called periodic if $f^n(a) = a$ for some positive integer n. The minimal n is called period of a, and $\{a, f(a), \dots, f^{n-1}(a)\}$ is called a periodic cycle. If a is an n-periodic point, we define $\lambda = (f^n)'(a)$ to be the multiplier of a. The point a is called attracting, repelling and neutral when $|\lambda| < 1$, $|\lambda| > 1$ and $|\lambda| = 1$ respectively. Thus we can define the Julia set of any entire function f as the closure of repelling periodic points. Moreover $J(f)$ is a perfect and completely invariant set. Standard references are [2] and [9] for rational functions and [6] for transcendental entire functions.

A point $a \in \mathbb{C}$ is called an asymptotic value of the entire function f if there is a curve $\gamma \subset \mathbb{C}$ tending to ∞ such that $f(z) \to a$ as $z \to \infty$ and $z \in \gamma$. If $f'(c) = 0$, then c is called a critical point of f and $f(c)$ is called a critical value. Let

$$\operatorname{sing} f^{-1} = \{\text{all critical values of } f\} \cup \{\text{all asymptotic values of } f\}$$

be the set of singular points of f. Set

$$S_q = \{f;\ f \text{ is entire and } \operatorname{sing} f^{-1} \text{ contains } q \text{ points}\}, \qquad S = \bigcup_{q=1}^{\infty} S_q.$$

The S is called the critically finite family. We call two entire functions $f(z)$ and $g(z)$ topologically equivalent in $\mathbb{C}$ if there are homeomorphisms ϕ, $\psi: \mathbb{C} \to \mathbb{C}$ such that $\psi \circ g = f \circ \phi$, and write $f \sim g$. For any nonconstant $g \in S_q$, set

$$M_g = \{f;\ f \text{ is entire and } f \sim g\}.$$

Clearly M_g is a subset of S_q. Moreover M_g [6, §3], is a $(q+2)$-dimensional complex analytic manifold and the topology of M_g is locally equivalent to the

[1] Reprinted from *Julia sets and bifurcation diagrams for exponential maps*, by R. Devaney, Bull. Amer. Math. Soc. **11** (1984), pp. 167–171 by permission of the American Mathematical Society.

topology of uniform convergence on compact subsets of $\mathbb{C}$. In the sequel, we write a given submanifold of M_g as simply M for some fixed $g \in S$.

Let M be such a submanifold. We define a multi-valued analytic function $\alpha_p: M \to \mathbb{C}$ as the set of solutions to the equation $f^p(\alpha) = \alpha$ (see [6] for full discussions). By [6, Theorem 2], the function α_p has only algebraic singularities. Let $\alpha_{p,i}(f)$ be a branch of α_p. Set

$$N_p = \{f \in M;\ (f^p)'(\alpha_{p,i}(f)) \text{ for some } i\}, \qquad \Sigma = \overline{M \backslash N},$$

$$\lambda_p(f) = (f^p)'(\alpha_p(f)), \qquad N = \bigcup_{p=1}^{\infty} N_p.$$

Clearly $\lambda_p(f)$ is a multi-valued analytic function. We call an entire function $f_0 \in M$ J-stable in M if for all $f \in M$ which are close enough to f_0, we have that $f_0|J(f_0)$ and $f|J(f)$ are topologically conjugate, i.e. there exists a homeomorphism $h_f: J(f_0) \to J(f)$ such that

$$h_f \circ f_0(z) = f \circ h_f(z), \qquad \text{for all } z \in J(f_0) \tag{1}$$

and h_f depends continuously on f under the topology of uniform convergence on compact sets, $h_{f_0} = \mathrm{id}$ and h_f is analytic as f ranges in M. We call an entire function $f_0 \in M$ structurally stable in M if for all $f \in M$ which are close enough to f_0, we have that f_0 and f are topologically conjugate in the whole complex plane and the conjugating homeomorphism depends continuously on f.

Definition. A holomorphic motion of a set $A \subset \mathbb{C}$ over U (originating at f_0) is a map $h: U \times A \to \mathbb{C}$ satisfying the following conditions:

(1) The map $h(f, z)$ is analytic in f for every $z \in A$,

(2) The map $h_f: z \mapsto h(f, z)$ is injective for every $f \in U$,

(3) $h_{f_0} = \mathrm{id}$.

2. Statement of results

Let $\lambda_0 \in \mathbb{C}\backslash\{0\}$ and δ be any positive real number.

$$O_\delta(\lambda_0) = \{\lambda;\ |\lambda - \lambda_0| < \delta\},\ \ E_\lambda(z) = \lambda e^z \text{ and } g_n(\lambda) = E_\lambda^n(0),$$

where E_λ^n is the n-fold iterate of $E_\lambda(z)$ and δ is so small that $O_\delta(\lambda_0) \cap \{0\} = \varnothing$. One important question concerning complex dynamical systems (see, for example, [3, 6]) asks:

Is there an open set of λ in the λ-plane for which $J(E_\lambda) = \mathbb{C}$? (Or, is $J(E_\lambda)$ structurally stable whenever $J(E_\lambda) = \mathbb{C}$?)

Based on the classification of Sullivan for Fatou set, it is known that (e.g. Baker and Rippon [1], Devaney [3]) that

Theorem A. *Let* $\lambda_0 \in \mathbb{C}\backslash\{0\}$.

($*$) *If* $g_n(\lambda_0) \to \infty$ $(n \to \infty)$, *then* $J(E_{\lambda_0}) = \mathbb{C}$.

($**$) *If* $\lambda_0 = k\pi i$ $(k \in \mathbb{Z})$, *then* $J(E_{\lambda_0}) = \mathbb{C}$.

In the sequel we say that λ satisfies condition ($*$) if $g_n(\lambda_0) \to \infty$ as $n \to \infty$ and condition ($**$) if $\lambda = k\pi i$, where $k \in \mathbb{Z}$.

Remark. In [6, Theorem 8], Eremenko and Lyubich generalized Theorem A to critically finite families. They proved that if the orbits of all singular points of $f \in S$ land on cycles or tend to ∞ then $J(f) = \mathbb{C}$.

It is known that E_λ $(\lambda > 1/e)$ is not structurally stable [4] and [11]. In this paper we first prove some equivalent propositions of J-stability for the family of critically finite entire functions and specialize to the family E_λ, and then prove E_λ is not structurally stable when λ satisfies condition $(*)$ or condition $(**)$. More precisely, we have

Theorem 1. *Let M be a submanifold of an M_g as above. Then the following statements are equivalent:*

(a) *The period of the longest attracting cycle of $f \in M$ is bounded uniformly in M.*

(b) *For each $f \in M$, $f \in \Sigma$.*

(c) *M is J-stable.*

(d) *Let $1 \le i \le q$, and choose $c_i(f) \in \operatorname{sing} f^{-1}$, and suppose that $c_i(f)$ is analytic in M and $\mathscr{F} = \{f^n(c_i(f))\}_{n=1}^{\infty}$ is an analytic family. Then $\mathscr{F}$ is a normal family.*

Corollary 2. *Let $\delta > 0$ be fixed, then the following statements are equivalent.*

(a) *The period of the longest attracting cycle of E_λ is bounded uniformly in $O_\delta(\lambda_0)$.*

(b) *For each $\lambda \in O_\delta(\lambda_0)$, $E_\lambda \in \Sigma$.*

(c) *For any $\lambda \in O_\delta(\lambda_0)$, E_λ is J-stable in $O_\delta(\lambda_0)$.*

(d) *$\{g_n(\lambda)\}$ is normal in $O_\delta(\lambda_0)$.*

Theorem 3. *If λ_0 satisfies the condition $(*)$, then E_{λ_0} is not J-stable.*

Corollary 4. *Let λ_0 satisfy condition $(*)$. Then given $\varepsilon > 0$, there exists a λ_* such that $|\lambda_0 - \lambda_*| < \varepsilon$ and $J(E_{\lambda_*}) \neq \mathbb{C}$; hence E_{λ_0} is not structurally stable.*

Theorem 5. *If λ_0 satisfies the condition $(**)$, then E_{λ_0} is not J-stable.*

Corollary 6. *let λ_0 satisfy condition $(**)$, then given $\varepsilon > 0$, there exists a λ_* such that $|\lambda_0 - \lambda_*| < \varepsilon$ and $J(E_{\lambda_*}) \neq \mathbb{C}$; hence E_{λ_0} is not structurally stable.*

Remark. It seems that we cannot prove the most general case when 0 is preperiodic by using the methods in this paper, although we may generalize Corollary 6 a little by adding some restrictions.

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3. Proofs

Proof of Theorem 1. We prove this theorem by combining the work of C. McMullen [8] for rational mappings with that of A. Eremenko and M. Lyubich [6] for critically finite entire functions. Here we use the following procedure (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (c) Suppose the period of the longest attracting cycle of any $f \in M$ is bounded uniformly in M. Then there exists an integer p such that all k-periodic cycles of any $f \in M$ are repelling whenever $k \ge p$. In fact, suppose there is an f which has a k-neutral cycle; i.e. $|\lambda_{k,i}(f)| = 1$, for some branch

$\lambda_{k,i}$ of λ_k. Since $\lambda_{k,i}(f)$ is a nonconstant analytic function [6, Lemma 6], there is an $f_1 \in M$ which is close to f with $|\lambda_k(f_1)| < 1$. So f_1 has an attracting cycle with period $k \geq p$, and this contradicts (a). Now for any $f_0 \in M$, let U be a simply-connected neighborhood of f_0 in M. For any $k \geq p$, all branches $\alpha_{k,i}$ of α_k may be chosen, in a natural way, as single-valued analytic functions in U since $f_0 \notin N_k$. Hence, as in [6, Theorem 9],

$$h_f\colon U \times \mathrm{Per}_p(f_0) \to \mathbb{C} \quad \text{with } h_f(\alpha_{k,i}(f_0)) = \alpha_{k,i}(f)$$

defines a holomorphic motion of the set of periodic points with period greater than p, which we write as $\mathrm{Per}_p(f_0)$, over U. (We denote the set of all periodic points of f by $\mathrm{Per}(f)$). In fact, clearly $h_{f_0} = \mathrm{id}$ and h_f is analytic in f as f ranges in U since $\alpha_{k,i}(f)$ is analytic. We next show that h_f is injection for any fixed $f \in U$. If not so, there exist $f_* \in U$, $\alpha_{s,i}(f_0)$, $\alpha_{t,j}(f_0)$ and $\alpha_{s,i}(f_0) \neq \alpha_{t,j}(f_0)$ such that $h_{f_*}(\alpha_{s,i}(f_0)) = \alpha_{s,i}(f_*) = \alpha_{t,j}(f_*) = h_{f_*}(\alpha_{t,j}(f_0))$. Set $F_{st}(f, z) = f^{st}(z) - z$. So $F_{st}(f_*, \alpha_{s,i}(f_*)) = F_{st}(f_*, \alpha_{t,j}(f_*)) = 0$ and $\frac{\partial}{\partial z}F_{st}(f_*, \alpha_{s,i}(f_*)) = (f_*^{st})'(\alpha_{s,i}(f_*)) - 1 \neq 0$ and $\frac{\partial}{\partial z}F_{st}(f_*, \alpha_{t,j}(f_*)) \neq 0$ since $f_* \in U$. Therefore applying the implicit function theorem to $F_{st}(f, z)$ at $(f_*, \alpha_{s,i}f_*)) = (f_*, \alpha_{t,j}(f_*))$, we see that by the uniqueness of the solution to the equation $F_{st}(f, z) = 0$, there is a neighborhood of f_*, say U', such that $\alpha_{s,i}(f) = \alpha_{t,j}(f)$, for $f \in U'$. Since these functions are analytic in U, they are equal to each other in U. Thus $\alpha_{s,i}(f_0) = \alpha_{t,j}(f_0)$, a contradiction.

It follows that the motion $h_f(z)$ is a holomorphic motion of $\mathrm{Per}_p(f_0)$ over U. By the λ-lemma [7], this motion can be extended to a holomorphic motion on $\overline{\mathrm{Per}_p(f_0)}$. However the set $\mathrm{Per}(f_0)\setminus \mathrm{Per}_p(f_0)$ has no limit point; otherwise we would have $f^n(z) \equiv z$, for some $n < p$. Hence $\mathrm{Per}_p(f_0)$ is dense in $J(f_0)$ since $\mathrm{Per}(f_0)$ is dense in $J(f_0)$ and $J(f_0)$ is perfect. Thus $\overline{\mathrm{Per}_p(f_0)} = J(f_0)$, and h_f provides the conjugation which shows the J-stability of f_0. Hence M is J-stable since f_0 was arbitrary.

To prove that (d) follows from (c), we need the following lemmas:

Lemma 1. *Let M be J-stable. If there exists $f_0 \in M$ such that the forward orbit of some $c_{i_0}(f_0) \in \operatorname{sing} f_0^{-1}$ meets a repelling cycle of f_0, then for any $f \in M$, the forward orbit of $c_{i_0}(f)$ under f must meet a repelling cycle of f.*

Proof. Suppose the forward orbit of some $c_{i_0}(f_0)$ meets a repelling cycle $\{f_0^m(z_0)\}_{m=0}^{p-1}$; i.e. there exist $0 \leq n_0 \leq p-1$ and m such that $f_0^{n_0}(z_0) = f_0^m(c_{i_0}(f_0))$, thus $c_{i_0}(f_0) \in J(f_0)$. Since M is J-stable, there is a neighborhood $U \subset M$ of f_0 such that for any $f \in U$, there exists a homeomorphism h_f such that (1) holds. Hence we have that

$$\{h_f(z_0), f \circ h_f(z_0), \dots, f^{p-1} \circ h_f(z_0)\}$$

is a repelling cycle of f. Furthermore the forward orbit of $h_f(c_{i_0}(f_0))$ under f meets this repelling cycle since

$$f^m \circ h_f(c_{i_0}(f_0)) = h_f \circ f_0^m(c_{i_0}(f_0)) = h_f \circ f_0^{n_0}(z_0) = f^{n_0} \circ h_f(z_0), \qquad f \in U.$$

Moreover both sides of this equality are analytic in M, so it holds in all of M. In other words, for any $f \in M$ the forward orbit of $h_f(c_{i_0}(f_0))$ meets a repelling cycle of f.

It remains to prove that $h_f(c_{i_0}(f_0)) = c_{i_0}(f)$ for $f \in M$, when $c_{i_0}(f_0) \in J(f_0)$. Since $c_{i_0}(f_0) \in J(f_0)$, there exists a sequence $\{a_k(f_0)\} \subset \mathrm{Per}_p(f_0)$ such that $a_k(f_0) \to c_{i_0}(f_0)$ $(k \to \infty)$. Moreover, we can assume from the fact that f_0 is J-stable that there are a neighborhood of f_0, say U, and $\{a_k(f)\} \subset \mathrm{Per}_p(f_0)$ such that $a_k(f) \to c_{i_0}(f)$ for $f \in U$. By our definition of h_f, we have that $h_f(a_k(f_0)) = a_k(f)$, hence $h_f(c_{i_0}(f_0)) = c_{i_0}(f)$ So the lemma is proved. □

Lemma 2. *Let* $c_i(f_0) \in \operatorname{sing} f_0^{-1}$. *Then there exists a repelling cycle with period greater than* 3 *of* f_0 *such that the forward orbit of* $c_i(f_0)$ *does not meet this cycle.*

Proof. Take any repelling cycle $\{f_0^k(z_0)\}_{k=0}^{p_0-1}$ $(p_0 \geq 3)$ of f_0. If the forward orbit of $c_i(f_0)$ does not hit this cycle, then the lemma is proved. Otherwise, noting f_0 has infinitely many repelling cycles, we can find another repelling cycle (period greater than 3) which is disjoint on the orbit of $c_i(f_0)$. Thus the lemma is completely proved. □

(c) ⇒ (d) Suppose M is J-stable. For any $1 \leq i \leq q$ and for fixed $f_0 \in M$, let $c_i(f_0) \in \operatorname{sing} f_0^{-1}$. Then by Lemma 2, there exists a repelling cycle $\{f_0^k(z_*)\}_{k=0}^{p_0-1}$ $(p_0 \geq 3)$ of f_0 such that the forward orbit of $c_i(f_0)$ under f_0 does not meet this repelling cycle; i.e. $\{f_0^n(c_i(f_0))\}_{n=0}^{\infty} \cap \{f_0^k(z_*)\}_{k=1}^{p_0-1} = \varnothing$. Thus we see from the proof of Lemma 1, $\{f^n(c_i(f))\}_{n=1}^{\infty}$ must avoid $\{h_f(f_0^k(z_*))\}_{k=0}^{p-1}$, for all $f \in M$, where h_f is from Lemma 1. Set

$$g_n^i(f) = \frac{f^n(c_i(f)) - h_f(z_*)}{f^n(c_i(f)) - h_f(f_0(z_*))} \frac{h_f(f_0^2(z_*)) - h_f(f_0(z_*))}{h_f(f_0^2(z_*)) - h_f(z_*)}. \tag{2}$$

So $\{g_n^i(f)\}_{n=1}^{\infty}$ does not hit 0, 1, ∞ (since h_f is one-to-one for each fixed $f \in M$), and $g_n^i(f)$ is holomorphic in M. By Montel's theorem $\{g_n^i(f)\}_{n=1}^{\infty}$ is normal. Moreover set $a(f) = [h_f(f_0^2(z_*)) - h_f(f_0(z_*))][h_f(f_0^2(z_*)) - h_f(z_*)]^{-1}$, it follows from the fact $h_f(z_*) \neq h_f(f_0(z_*))$ that $g_n^i(f) \neq a(f)$. Thus solving for f^n in (2), we have that

$$f^n(c_i(f)) = \frac{g_n^i(f) h_f(f(z_*)) - h_f(z_*) a(f)}{g_n^i(f) - a(f)}.$$

Suppose that $g_{n_j}^i(f)$ tends to $b(f)$ on a compact subset of M. Then we have that either $b(f) \equiv a(f)$ or $b(f) \neq a(f)$, otherwise $g_{n_j}^i(f) - a(f) = 0$ has solutions for all large j. It follows that $\mathscr{F}$ is normal. Thus (d) is proved.

(d) ⇒ (a) Suppose that the holomorphic family $\{f^n(c_i(f))\}_{n=1}^{\infty}$ is normal in M for each i. Let $F_i(f)$ denote thefunction which is the uniform limit of a subsequence of $\{f^n(c_i k(f))\}_{n=1}^{\infty}$ in some compact subset of M. Now suppose for some $f_0 \in M$, f_0 has an attracting cycle with some order k, say $\{f_0^j(z_0)\}_{j=0}^{k-1}$. (If not so, (a) is proved.) Thus this cycle must attract some $F_{i_0}(f_0)$, since each attracting basin contains a singular point of f_0, (e.g. see [6, §5]); i.e. $f_0^k(F_{i_0}(f_0)) = F_{i_0}(f_0)$. Hence $F_{i_0}(f) \neq \infty$. Applying the implicit function theorem to $\phi(f, z) = f^k(z) - z$ at the point (f_0, z_0) and noting the continuity of $\lambda_k(f, z) = (f^k)'(z)$, we find a neighborhood $U(f_0)$ such that any $f \in U(f_0)$ has an attracting cycle with period k. Since each attracting basin of f contains a singular point of f and the subsequence of the forward orbits of the singular point uniformly approaches the attracting cycle, we have that $f^k(F_{i_0}(f)) = F_{i_0}(f)$, for any $f \in U(f_0)$. Hence it holds in M. Thus $c_{i_0}(f)$ (the i_0th singular point of f) is never attracted to a cycle with period greater than k for all $f \in M$. Therefore if there is another $f_1 \in M$ which has an attracting cycle with period $k_1 > k$, then there is $c_{i_1}(f)$ $(i_1 \neq i_0)$ such that $c_{i_1}(f)$ is attracted by a cycle with period no than k_1. However there are only finitely many singular points, so the longest length of an attracting cycle must be uniformly bounded.

Remark. Eremenko and Lyubich [6, Theorem 5] proved that if $f \in S_q$, then the number of attracting cycles is not greater than q. Hence the longest length of attracting cycle of f is bounded.

(c) $\Rightarrow$ (b) Suppose f_0 is J-stable in M. Then there is a $U(f_0)$ in M such that the number of repelling periodic cycles of any f in $U(f_0)$ with period p is constant, since $f_0|J(f_0)$ is topologically conjugate to $f|J(f)$ for any $f \in U(f_0)$ and for any integer p. Now we suppose that (b) is not true, i.e. $f_0 \in N = \overline{\bigcup_{p=1}^{\infty} N_p}$. If $f_0 \in N_p$, for some p, then $\lambda_{p,i}(f_0) = 1$ for some branch of λ_p. Since $\lambda_{p,i}(f)$ is a nonconstant analytic function [6, Lemma 6], there is a f such that $f \in M$, f arbitrarily close to f_0 such that the number of repelling cycles of f with period p is greater than that of f_0. This contradicts the assumption that the number of repelling cycles with period p is constant in $U(f_0)$. Now we consider the case that $f_0 \in N$ but $f_0 \notin N_p$ for any p. Thus there exists a sequence f_n such that $f_n \in \bigcup_{p=1}^{\infty} N_p$ and f_n converges uniformly to f_0 on any compact subset. But the number of repelling cycles of order p of any f in $U(f_0)$ is constant for any fixed p. However we may take a $f_{n_0} \in N_{p_0} \cap U(f_0)$, so that $\lambda_{p_0,i}(f_n) = 1$. As above there exists $f_* \in U(f_0)$ such that the number of repelling cycles with period p_0 of f_* is greater than that of f_{n_0} which in turn is equal to that of f_0. This again contradicts the fact that the number of repelling cycles with period p is constant in $U(f_0)$. So we have proved the (b).

(b) $\Rightarrow$ (c) See [6, Theorem 9].

Thus Theorem 1 is proved completely. □

Proof of Corollary 2. We take $M = \{E_\lambda;\ \lambda \in \Omega_\delta(\lambda_0)\}$. As in §1, let $\psi = \mathrm{id}$ and $\phi = z + \log(\lambda_1/\lambda_2)$, we see that $E_{\lambda_1} \sim E_{\lambda_2}$ (for any λ_1, $\lambda_2 \in O_\delta(\lambda_0)$). Hence M is a submanifold of S_1. Furthermore the exponential family has only one asymptotic value 0; i.e. $\mathrm{sing}\, E_\lambda^{-1} = \{0\}$ for all $\lambda \in O_\delta(\lambda_0)$. Hence $f^n(c(f)) = E_\lambda^n(0) = g_n(\lambda)$. Thus the corollary follows from Theorem 1. □

Proof of Theorem 3. If E_{λ_0} is J-stable, then there exists $\delta_0 > 0$ such that $M_0 = \{E_\lambda;\ \lambda \in O_{\delta_0}(\lambda_0)\}$ is J-stable. By Corollary 2, $\{g_n(\lambda)\}$ is normal in $O_{\delta_0}(\lambda_0)$. By hypothesis, $g_n(\lambda_0) = E_{\lambda_0}^n(0)$ tends to infinity, thus $g_n(\lambda)$ converges uniformly to infinity in $O_{\delta_1}(\lambda_0)$, where $\delta_1 \leq \delta_0$. Without loss of generality we may assume $\delta = \delta_0 = \delta_1$. Moreover, from the definition of $g_n(\lambda)$ we have that

$$g_{n+1}(\lambda) = \lambda e^{g_n(\lambda)}. \tag{3}$$

since g_{n+1} tends uniformly to infinity, (3) implies that $\mathrm{Re}\, g_n(\lambda)$ also tends uniformly to $+\infty$. Now we consider two cases by looking at $\{g_n'(\lambda_0)\}$.

Case One: $\{g_n'(\lambda_0)\}$ is unbounded.

Choose a subsequence n_k with $\{g_{n_k}'(\lambda_0)\}$ tending to infinity $(k \to \infty)$. Set $h_m(\lambda) = g_n(\lambda) + \log \lambda$, for any fixed branch of logarithm. We may assume that each $h_n(\lambda)$ is analytic in $O_\delta(\lambda_0)$. Furthermore, given $M > 1$, there is a N_1 such that $|h_{n_k}'(\lambda_0)|\delta \geq 8M$ $(k \geq N_1)$. By applying Bloch's theorem to $\{h_{n_k}(\lambda)\}_{k=N_1}^{\infty}$ in $O_\delta(\lambda_0)$, we obtain that there is an open set $D_{n_k} \subset O_\delta(\lambda_0)$ such that $h_{n_k}(D_{n_k})$ is a disk with $\operatorname{diam}(h_{n_k}(D_{n_k})) \geq \sqrt{3}|h'_{n_k}(\lambda_0)|\delta/2 \geq 4M$, $(k \geq N_1)$. Thus there exists $\lambda^*_{n_k} \in D_{n_k} \subset O_\delta(\lambda_0)$ and an integer j_{n_k} such that $\operatorname{Im}(h_{n_k}(\lambda^*_{n_k})) = (2j_{n_k} + 1/2)\pi$, $(k \geq N_1)$. So, if $\lambda^*_{n_k} = r^*_{n_k} e^{i\theta^*_{n_k}}$, we may arrange that $\operatorname{Im}(g_{n_k}(\lambda^*_{n_k})) = (2j_{n_k} + 1/2)\pi - \theta^*_{n_k}$, $(k \geq N_1)$. Hence, by (3),

$$\begin{aligned}\operatorname{Re} g_{n_k+1}(\lambda^*_{n_k}) &= \operatorname{Re}\{r^*_{n_k} e^{\operatorname{Re} g_{n_k}(\lambda^*_{n_k}) + i\{\operatorname{Im} g_{n_k}(\lambda^*_{n_k}) + \theta^*_{n_k}\}}\} \\ &= r^*_{n_k} e^{\operatorname{Re} g_{n_k}(\lambda^*_{n_k})} \cos(\operatorname{Im} g_{n_k}(\lambda^*_{n_k}) + \theta^*_{n_k}) \\ &= 0, \qquad \text{for all } k \geq N_1.\end{aligned}$$

This contradicts the uniform convergence of $\{\operatorname{Re} g_n(\lambda)\}$ to infinity in $O_\delta(\lambda_0)$, so E_{λ_0} is not J-stable and the theorem is proved in this case.

Case Two: $\{g'_n(\lambda_0)\}$ is a bounded set. First we prove the following two claims.

Claim 1. *If $\{g'_n(\lambda_0)\}$ is bounded, then we have that $g'_n(\lambda_0) \to -1/\lambda_0$, as $n \to \infty$.*

Proof. If there is a subsequence $\{g'_{n_k}(\lambda_0)\}$ with limit $a \neq -1/\lambda_0$, then since $g'_{n_k+1}(\lambda_0) = e^{g_{n_k}(\lambda_0)}(1 + \lambda_0 g'_{n_k}(\lambda_0))$, $\operatorname{Re} g_{n_k}(\lambda_0) \to +\infty$ and $1 + \lambda_0 g'_{n_k}(\lambda_0) \to 1 + \lambda_0 a \neq 0$, we must have that $g'_{n_k+1}(\lambda_0) \to \infty$, as $k \to \infty$. This contradicts the boundedness of $\{g'_n(\lambda_0)\}$.

Claim 2. *If $\{g'_n(\lambda_0)\}$ is bounded, then there exists $\delta > 0$ such that $g'_n(\lambda) \to -1/\lambda$ in $O_\delta(\lambda_0)$, as $n \to \infty$.*

Proof. If there is a sequence λ_j in $O_\delta(\lambda_0)$ with $\lambda_j \to \lambda_0$ and $\{g'_n(\lambda_j)\}_{n=1}^{\infty}$ is unbounded for each j, then by Case One, for each j, E_{λ_j} cannot be J-stable. It follows from the definition of J-stability of E_{λ_0} that E_{λ_0} is not J-stable. Hence there exists $\delta_2 < \delta$ such that $\{g'_n(\lambda)\}_{n=1}^{\infty}$ is bounded for each fixed $\lambda \in O_{\delta_2}(\lambda_0)$. By Claim 1 we must have $g'_n(\lambda) \to -1/\lambda$ in $O_{\delta_2}(\lambda_0)$. Again we assume $\delta_2 = \delta$.

Now we want to get contradiction under the assumptions that $g'_n(\lambda)$ tends to $-1/\lambda$ in $O_\delta(\lambda_0)$ and that $g_n(\lambda)$ and $\operatorname{Re} g_n(\lambda)$ uniformly go to infinity and positive infinity respectively in $O_\delta(\lambda_0)$. Again we consider two cases.

Case (2.1). The convergence in Claim 2 is not uniform in any neighborhood of λ_0.

Case (2.2). The convergence in Claim 2 is uniform in some neighborhood of λ_0.

Case (2.1): Since $g'_n(\lambda)$ converges to $-1/\lambda$ and does not converge uniformly to $-1/\lambda$ in any $O_\delta(\lambda_0)$, there exists a subsequence $\lambda_{n_k} \in O_\delta(\lambda_0)$ with $\lambda_{n_k} \to \lambda_0$ such that $g'_{n_k}(\lambda_{n_k})$ tends to a constant $b \neq -1/\lambda_0$. If $b \neq \infty$, then $g'_{n_k+1}(\lambda_{n_k}) = e^{g_{n_k}(\lambda_{n_k})}(1 + \lambda_{n_k} g'_{n_k}(\lambda_{n_k}))$ tends to infinity since $e^{g_{n_k}(\lambda_{n_k})} \to \infty$ and $(1 + \lambda_{n_k} g'_{n_k}(\lambda_{n_k})) \to (1 + b\lambda_0) \neq 0$. So without loss of generality, we may assume $b = \infty$, i.e. $g'_{n_k}(\lambda_{n_k}) \to \infty$. Thus there exist $\tilde{\lambda} \in \{\lambda_{n_k}\}_{k=1}^{\infty}$ and N_0 such that

$$|g'_{N_0}(\tilde{\lambda})| \geq 2/|\tilde{\lambda}| \quad \text{and} \quad |g_n(\tilde{\lambda})| \geq 2, \quad (n > N_0).$$

Therefore we have, for all $n > N_0$,

$$\begin{aligned}|\tilde\lambda g_n'(\tilde\lambda)| &= |g_n(\tilde\lambda) + g_n(\tilde\lambda)\tilde\lambda g_{n-1}'(\tilde\lambda)| \\ &= |g_n(\tilde\lambda) + g_n(\tilde\lambda)g_{n-1}(\tilde\lambda) + g_n(\tilde\lambda)g_{n-1}(\tilde\lambda)\tilde\lambda g_{n-2}'(\tilde\lambda)| \\ &= |g_n(\tilde\lambda) + g_n(\tilde\lambda)g_{n-1}(\tilde\lambda) + \cdots + g_n(\tilde\lambda)\cdots g_{N_0+1}(\tilde\lambda) \\ &\qquad + g_n(\tilde\lambda)\cdots g_{N_0+1}(\tilde\lambda)\tilde\lambda g_{N_0}'(\tilde\lambda)| \\ &\geq 2|g_n\cdots g_{N_0+1}(\tilde\lambda)| - |g_n\cdots g_{N_0+1}(\tilde\lambda)| - \cdots - |g_n g_{n-1}(\tilde\lambda)| - |g_n(\tilde\lambda)| \\ &\geq |g_n(\tilde\lambda)| \to \infty, \qquad \text{as } n\to\infty.\end{aligned}$$

This is impossible since by Claim 2, $\tilde\lambda g_n'(\tilde\lambda) \to -1$. Hence we have the theorem in Case (2.1).

Case (2.2): In this case we know that $g_n(\lambda)$ and $\operatorname{Re} g_n(\lambda)$ uniformly tend to ∞ and $+\infty$ respectively, and $g_n'(\lambda)$ uniformly tends to $-1/\lambda$ in $O_\delta(\lambda_0)$. Hence $g_n(\lambda) - g_n(\lambda_0)$ uniformly tends to $\log(\lambda_0/\lambda)$ in $O_\delta(\lambda_0)$. We may assume δ so small that $\log(\lambda_0/\lambda)$ is univalent in $O_\delta(\lambda_0)$. In the sequel, we regard N_0 as a sufficiently large number whose value each time may not be the same. By the Koebe's distortion theorem [10], for $n > N_0$ and $\lambda \in O_\delta(\lambda_0)$, we have

$$\begin{aligned}&|g_n'(\lambda_0)||\lambda-\lambda_0|(1+\lambda-\lambda_0|\delta^{-1})^{-2} \\ &\quad\leq |g_n(\lambda)-g_n(\lambda_0)| \leq |g_n(\lambda_0)||\lambda-\lambda_0|(1-|\lambda-\lambda_0|\delta^{-1})^{-2}.\end{aligned} \tag{5}$$

Since $g_n'(\lambda_0) \to -1/\lambda_0$, we have that $g_n(O_\delta(\lambda_0))$ $(n > N_0)$ contains a disk with center $g_n(\lambda_0)$ whose radius η_0 is uniformly bounded from 0.

Set $\lambda_0 = r_0 e^{i\theta_0}$ with $0 \leq \theta_0 < 2\pi$, let $\gamma(\theta_0)$ be the radial segment, i.e.

$$\gamma(\theta_0) = \{\lambda;\, \lambda = re^{i\theta_0} \in O_\delta(\lambda_0),\, r_0-\delta \leq r \leq r_0+\delta\},$$

We obtain from (5) that

$$\min\{|g_n((r_0+\delta)e^{i\theta_0}) - g_n(\lambda_0)|,\, |g_n((r_0-\delta)e^{i\theta_0}) - g_n(\lambda_0)|\} > \delta/8|\lambda_0| \equiv r_*, \tag{6}$$

for all large n. Since $g_n(\lambda) - g_n(\lambda_0) \to \log(\lambda_0/\lambda)$ we have that

$$\operatorname{Re} g_n((r_0+\delta)e^{i\theta_0}) < \operatorname{Re} g_n(\lambda_0) < \operatorname{Re} g_n((r_0-\delta)e^{i\theta_0}), \quad (n > N_0). \tag{7}$$

Now $\log(\lambda_0/\lambda)$ maps $\gamma(\theta_0)$ to a horizontal line segment on real axis, so that $g_n(\lambda) = \log(\lambda_0/\lambda) + g_n(\lambda_0) + \varepsilon_n(\lambda)$ (where $\varepsilon_n(\lambda)$ tends uniformly to zero) maps $\gamma(\theta_0)$ to a curve segment Δ_n which for large n is asymptotic to a horizontal segment line passing through $g_n(\lambda_0)$. However since the exponential function maps a horizontal line to a ray starting at the origin, the equality $g_{n+1}(\lambda) = re^{i\theta_0}e^{g_n(\lambda)}$, $\lambda \in \gamma(\theta_0)$, tells us that the image $g_{n+1}(\gamma(\theta_0))$ is a curve segment γ_{n+1} which tends to the line segment contained in the ray $\Gamma_{n+1} = \{te^{i(\operatorname{Im}_{g_n}(\lambda_0)+\theta_0)},\ t > 0\}$ $(n > N_0)$. Hence the two curve segments Δ_n and γ_n must be close when $n > N_0$. Since $e^{Re_{g_n}(\lambda)}$ is large for $n > N_0$, this means that if $\lambda \in \gamma(\theta_0)$ then $\operatorname{Im} g_n(\lambda) + \theta_0 = 2k_n\pi + \varepsilon_n$, where $k_n = k_n(\theta_0)$ is an integer and $\varepsilon_n \to 0$. Therefore $(\operatorname{Im} g_n(\lambda) + \theta_0) \pmod{2\pi}$ converges uniformly to zero in $\gamma(\theta_0)$ since $\operatorname{Im} g_n(\lambda) + \operatorname{Im} g_n(\lambda_0)$ tends uniformly to 0 in $\gamma(\theta_0)$ and $(\operatorname{Im} g_n(\lambda) + \theta_0) \pmod{2\pi}$ tends to zero.

Moreover using (6), (7), the fact that $\operatorname{Im} g_n(\lambda_0) \to -\theta_0 \pmod{2\pi}$ and that Δ_n is close to a segment of γ_n, we may take a point $\lambda_n \in \gamma(\theta_0)$ such that $\operatorname{Im}_{g_n}(\lambda_n) + \theta_0 = 0 \pmod{2\pi}$. Thus $\operatorname{Im} g_{n+1}(\lambda_n) = 0$. So if $\theta_0 \neq 0$, this contradicts the fact that $\operatorname{Im} g_n(\lambda) \pmod{2\pi}$ tends uniformly to $-\theta_0$ in $\gamma(\theta_0)$. However if $\theta_0 = 0$, then λ_0 must be real. Moreover $E^n_{\lambda_0}(0) \to \infty$ implies that $\lambda_0 > 1/e$. It can be easily seen that $q'_n(\lambda_0)$ must tend to infinity, which has already been discussed in Case One in this case. Thus Theorem 3 is completely proved. □

Proof of Corollary 4. By Theorem 3, we know that E_{λ_0} is not J-stable. Hence there is a $\delta < \varepsilon$ such that the family $\{\lambda e^z;\ |\lambda - \lambda_0| < \delta\}$ is not J-stable. But J-stability is equivalent to structural stability since $J(E_{\lambda_0}) = \mathbb{C}$. It follows from Corollary 2(b) that there is a $\lambda_* \in O_\delta(\lambda_0)$ such that $E_{\lambda_*} \notin \Sigma$. Therefore $J(E_{\lambda_*}) \neq \mathbb{C}$ since $J(E_{\lambda_0}) = \mathbb{C}$.

Remark 1. By [3], we know that there are uncountable unions of curves tending to ∞ in the λ-plane such that for all λ except on the endpoints of these curves, $E^n_\lambda(0) \to \infty$.

Remark 2. If $\lambda_0 > 0$, then both $g_n(\lambda_0)$ and $g'_n(\lambda_0)$ are positive. So

$$g'_n(\lambda_0) = e^{g_{n-1}(\lambda_0)} + g_n(\lambda_0) g'_{n-1}(\lambda_0) \geq e^{g_{n-1}(\lambda_0)} \to \infty, \qquad (n \to \infty).$$

Hence by Theorem 3 (Case One) and Corollary 4, we have an alternate proof of [4 and 11].

Proof of Theorem 5. We first prove the case $\lambda_0 = 2k\pi i$ ($k \neq 0$ is a given integer). Then we have $E_{\lambda_0}(0) = 2k\pi i = E^2_{\lambda_0}(0)$; i.e. $g_2(\lambda_0) = \lambda_0 = g_1(\lambda_0)$, so, $g_n(\lambda_0) = \lambda_0$, $(n > 0)$. As in the proof of Theorem 3, if E_{λ_0} is J-stable, it follows from Corollary 2(d) that there exists $\delta > 0$ such that $\{g_m(\lambda)\} = \{E^m_\lambda(0)\}$ is normal in $O_\delta(\lambda_0)$. Hence there exists a subsequence $\{g_{m_j}(\lambda)\}$ which necessarily converges to a holomorphic function $g(\lambda)$ in $O_{\delta/2}(\lambda_0)$ since $g_n(\lambda_0) = \lambda_0$, for all $n > 0$. Thus $\{g'_{m_j}(\lambda_0)\}$ is bounded since $g'(\lambda_0)$ is finite. However from (3) we have that

$$\lambda g'_n(\lambda) = g_m(\lambda) + g_m(\lambda)\lambda g'_{m-1}(\lambda), \qquad \text{for all } m > 0. \tag{8}$$

Hence we obtain using (8) that $(\lambda_0 \neq 0)$,

$$\begin{aligned}\lambda_0 g'_m(\lambda_0) &= g_m(\lambda_0) + g_m(\lambda_0) g_{m-1}(\lambda_0) + \cdots \\ &\quad + g_m(\lambda_0) \cdots g_2(\lambda_0) + g_m(\lambda_0) \cdots g_2(\lambda_0) \lambda_0 g'_1(\lambda_0) \\ &= \lambda_0 + \lambda_0^2 + \cdots + \lambda_0^{m-1} + \lambda_0^m \\ &= \lambda_0(\lambda_0^m - 1)/(\lambda_0 - 1) \to \infty.\end{aligned}$$

This contradicts the boundedness of the $\{g'_{m_j}(\lambda_0)\}$, and Theorem 5 is proved in this case.

Next set $\lambda_0 = 2(k+1)\pi i$ (k a fixed integer). Then $E^2_{\lambda_0}(0) = -2(k+1)\pi i = E^3_{\lambda_0}(0)$. Set $a = E^2_{\lambda_0}(0)$. We have that $E^n_{\lambda_0}(0) = a$ and $g_n(\lambda_0) = a$, $(n \geq 2)$. Hence if the theorem is not true for this λ_0, then there exists $\delta > 0$ such that $\{g_m(\lambda)\} = \{E^m_\lambda(0)\}$ is normal in $O_\delta(\lambda_0)$. Therefore there exists a subsequence such that $\{g'_{m_j}(\lambda_0)\}$ is a bounded set since $g_n(\lambda_0) = a$ $(n \geq 2)$. However, again we have that

$$\begin{aligned}\lambda_0 g_m'(\lambda_0) &= g_m(\lambda_0) + g_m(\lambda_0)g_{m-1}(\lambda_0) + \cdots \\ &\quad + g_m(\lambda_0)\cdots g_2(\lambda_0) + g_m(\lambda_0)\cdots g_2(\lambda_0)\lambda_0 g_1'(\lambda_0) \\ &= a + a^2 + \cdots + a^{m-1}a^{m-1}\lambda_0 \\ &= \frac{a}{1-a} - a^m \frac{2-a}{1-a} \to \infty .\end{aligned}$$

Therefore as in the paragraph above we have a contradiction. Thus the theorem is completely proved. □

Proof of Corollary 6. The proof is quite similar to that of Corollary 4. □

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